Generalized Sampling and Infinite Dimensional Compressed Sensing

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Magnetic Resonance Imaging (MRI)

Let $\mathcal{F}$ denote the Fourier Transform. In particular,

$$(\mathcal{F}g)(\omega) = \int g(x)e^{-2\pi i\omega x}$$

Let

$$f = \mathcal{F}g.$$

We want to recover g (completely) from samples of f.

Compressed Sensing

Let $U \in \mathbb{C}^{n \times n}$, $x_0 \in \mathbb{C}^n$ and consider

$$y = Ux_0.$$

We want to recover x_0 from y. This is obvious if U is invertible and we know y.
What if we do not know y, but rather

$$P_\Omega y,$$

where P_Ω is the projection onto $\mathrm{span}\{e_j\}_{j\in\Omega}$ and $\Omega \subset \{1, \dots, n\}$ with $|\Omega| = m$ and Ω is randomly chosen.
Can we recover x_0 from $P_\Omega y$?

Magnetic Resonance Imaging (MRI)

Let U be the discrete Fourier Transform and x_0 be an image of the brain. The question is now: How to reconstruct x_0 from the measurement vector y. In particular, we have:

$x_0 =$ 

$$y = Ux_0,$$

Sparsity

Given $x_0 \in \mathbb{C}^n$ let

$$\Delta = \{k \in \mathbb{N} : \langle x_0, e_k \rangle \neq 0\}.$$

Want to find a strategy so that x_0 can be reconstructed from $P_\Omega U x_0$, where $|\Omega| = m$, with high probability. In particular we would like to know how large m must be as a function of n and $|\Delta|$.

Convex Optimization

Want to recover x_0 from $P_\Omega U x_0$ by finding

$$\inf_x \|x\|_{l^0}, \qquad P_\Omega U x = P_\Omega U x_0 \tag{1}$$

where $\|x\|_{l^0} = |\{j : x_j \neq 0\}|$ or

$$\inf_x \|x\|_{l^1}, \qquad P_\Omega U x = P_\Omega U x_0, \tag{2}$$

where $\|x\|_{l^1} = \sum_{j=1}^{n} |x_j|$. Note that (1) is a non-convex optimization problem and (2) is a convex optimization problem.

Theoretical Results

Theorem

(Candes, Romberg, Tao) Let $x_0 \in \mathbb{C}^n$ be a discrete signal supported on an unknown set Δ, and choose Ω of size $|\Omega| = m$ uniformly at random. For a given accuracy parameter M there is a constant C_M such that if

$$m \geq C_M \cdot |\Delta| \cdot \log(n)$$

then with probability at least

$$1 - \mathcal{O}(n^{-M}),$$

the minimizer to the problem (2) is unique and is equal to x_0 .

The Shannon Sampling Theorem

Suppose that

$$f = \mathcal{F}g, \qquad g \in L^2(\mathbb{R}),$$

and $\mathrm{supp}(g) \subset [-T, T]$ for some $T > 0$. If $\epsilon \leq \frac{1}{2T}$ (the Nyquist rate) then

$$f(t) = \sum_{k=-\infty}^{\infty} f(k\epsilon)\mathrm{sinc}\left(\frac{t + k\epsilon}{\epsilon}\right), \quad L^2 \text{ and unif. conv.,} \tag{3}$$

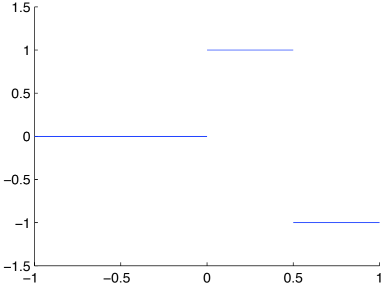
$$g = \epsilon \sum_{k=-\infty}^{\infty} f(k\epsilon)e^{2\pi i \epsilon k \cdot}, \qquad L^2 \text{ convergence.} \tag{4}$$

In practice, one forms the approximations

$$f_N = \sum_{k=-N}^{N} f(k\epsilon)\mathrm{sinc}\left(\frac{t + k\epsilon}{\epsilon}\right), \qquad g_N = \epsilon \sum_{k=-N}^{N} f(k\epsilon)e^{2\pi i \epsilon k \cdot}.$$

Example:

Let g be



The Shannon Sampling Theorem

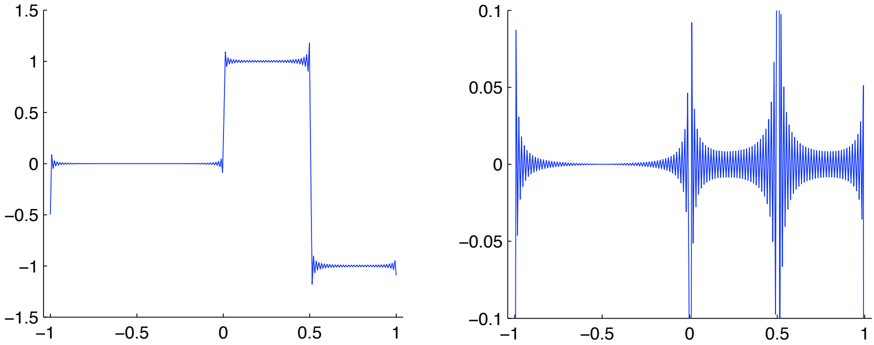


Figure: The figure displays g_N (left) as well as the error $g - g_N$ (right). $N = 128$

The Shannon Sampling Theorem

(i) g_N does NOT converge uniformly to g.

(ii) g_N converges VERY slowly to g in the L^2 norm.

The Discrete Problem

Note that

$$g_N = \epsilon \sum_{k=-N+1}^{N} f(k\epsilon) e^{2\pi \mathrm{i} \epsilon k \cdot}.$$

can be written as

$$y = U_{df} x, \qquad U_{df} \in \mathbb{C}^{2N \times 2N}, \quad y, x \in \mathbb{C}^{2N},$$

where y represents a vector of the sampled values of f, x represents a vector of the point wise values of g_N (on a equidistant grid on [-1,1]), and U_{df} is a scalar multiple of the discrete Fourier transform.

The Finite Dimensional Compressed Sensing Problem

If g is sparse in the Haar basis one could hope that

$$x_0 = V_{dw} x$$

is sparse, where V_{dw} is the discrete wavelet transform corresponding to the Haar wavelet. If that was the case we could make use of the Compressed Sensing framework and randomly sample a set $\Omega \subset \{1, \ldots, 2N\}$ of size $|\Omega| = m < 2N$ and try to reconstruct x_0 (and hence x) from the subsampled vector $P_\Omega y$ by finding a minimizer ξ to

$$\min_{\eta \in \mathbb{C}^n} \|\eta\|_{l^1} \ : \ P_\Omega U_{df} V_{dw}^{-1} \eta = P_\Omega y, \tag{5}$$

where P_Ω denotes the projection onto $\mathrm{span}\{e_j\}_{j\in\Omega}$, and hope that $\xi = x_0$ with high probability.

Finite Dimensional Compressed Sensing Results

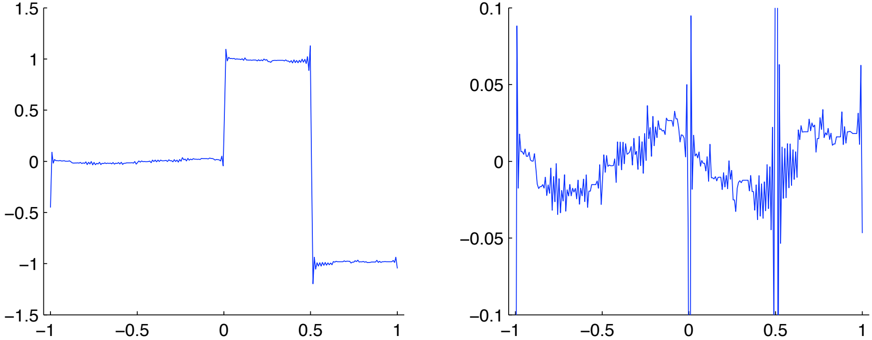


Figure: The left part displays the compressed sensing approximation $V_{dw}^{-1}\xi$ to g_N from solving (5) with $|\Omega| = 130$. The right part displays the error and $g - V_{dw}^{-1}\xi$.

Why Infinite Dimensions

(i) If we know that g is sparse (or has rapidly decaying coefficients) in the Haar basis (or any other basis) why are we using the (possibly) slowly converging approximation

$$g_N = \epsilon \sum_{k=-N+1}^{N} f(k\epsilon) e^{2\pi i \epsilon k \cdot},$$

when g_N is not sparse?

(ii) Why are we using the sparsifying transformation V_{dw} and solve (5) when we cannot do better than recovering g_N, as this is what we recover by solving (5) with $P_\Omega = I$ (i.e. full sampling)?

(iii) Why are we not trying to obtain the coefficients $\{\beta_k\}_{k \in \mathbb{N}}$ that have only a few non zeroes (or are rapidly decaying) in the expansion

$$g = \sum_{k=1}^{\infty} \beta_k \psi_k?$$

If we could reconstruct the β_js, we would recover g perfectly.

The Model

- Given a separable Hilbert space $\mathcal{H}$ with an orthonormal set $\{\varphi_k\}_{k\in\mathbb{N}}$.
- Given a vector

$$x_0 = \sum_{k=1}^{\infty} \beta_k \varphi_k, \qquad \beta = \{\beta_1, \beta_2, \dots\}.$$

- Suppose also that we are given a set of linear functionals $\{\zeta_j\}_{j\in\mathbb{N}}$ such that we can "measure" the vector x_0 by applying the linear functionals e.g. we can obtain $\{\zeta_j(x_0)\}_{j\in\mathbb{N}}$.

An Infinite System of Equations

With some appropriate assumptions on the linear functionals $\{\zeta_j\}_{j\in\mathbb{N}}$ we may view the full recovery problem as the infinite dimensional system of linear equations

$$\begin{pmatrix} \zeta_1(x_0) \\ \zeta_2(x_0) \\ \zeta_3(x_0) \\ \vdots \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \end{pmatrix}, \qquad u_{ij} = \zeta_i(\varphi_j), \tag{6}$$

where we will refer to $U = \{u_{ij}\}_{i,j\in\mathbb{N}}$ as the "measurement matrix".

What to Do?

- Abandon the finite section method.
- Use uneven section techniques from:
  A. C. Hansen. On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators. *J. Amer. Math. Soc., 24(1):81124, 2011.*

Solution I

If we for example have that U forms an isometry on $l^2(\mathbb{N})$ we could, for every $K \in \mathbb{N}$, compute an approximation $x = \sum_{k=1}^{K} \tilde{\beta}_k \varphi_j$ by solving

$$A \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \tilde{\beta}_3 \\ \vdots \\ \tilde{\beta}_K \end{pmatrix} = P_K U^* P_N \begin{pmatrix} \zeta_1(x_0) \\ \zeta_2(x_0) \\ \zeta_3(x_0) \\ \vdots \end{pmatrix}, \qquad A = P_K U^* P_N U P_K|_{P_K l^2(\mathbb{N})},$$

for some appropriately chosen $N \in \mathbb{N}$ (the number of samples). We would then get the following error:

$$\|x - x_0\|_{\mathcal{H}} \leq (1 + C_{K,N}) \|P_K^{\perp} \beta\|_{l^2(\mathbb{N})}, \quad \beta = \{\beta_1, \beta_2, \dots\},$$

Solution II

where, for fixed K, the constant $C_{K,N} \to 0$ as $N \to \infty$. Moreover, the constant $C_{K,N}$ is given explicitly by

$$C_{K,N} = \left\| (P_K U^* P_N U P_K|_{P_K l^2(\mathbb{N})})^{-1} P_K U^* P_N U P_K^{\perp} \right\|,$$

and hence we may find, for any $K \in \mathbb{N}$, the appropriate choice of $N \in \mathbb{N}$ (the number of samples) to get the desired error bound. In particular, this can be done numerically, by computing with different sections of the infinite matrix U.

The Generalized Sampling Theorem

Theorem

(Adcock,H'10) Let $\mathcal{F}$ denote the Fourier transform on $L^2(\mathbb{R}^d)$. Suppose that $\{\varphi_j\}_{j\in\mathbb{N}}$ is an orthonormal set in $L^2(\mathbb{R}^d)$ such that there exists a $T > 0$ with $\mathrm{supp}(\varphi_j) \subset [-T, T]^d$ for all $j \in \mathbb{N}$. For $\epsilon > 0$, let $\rho : \mathbb{N} \to (\epsilon\mathbb{Z})^d$ be a bijection. Define the infinite matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad u_{ij} = (\mathcal{F}\varphi_j)(\rho(i)). \tag{7}$$

Then, for $\epsilon \leq \frac{1}{2T}$, we have that $\epsilon^{d/2}U$ is an isometry.

Theorem

Also, set

$$f = \mathcal{F}g, \quad g = \sum_{j=1}^{\infty} \beta_j \varphi_j \in L^2(\mathbb{R}^N),$$

and let (for $l \in \mathbb{N}$)P_l denote the projection onto $\mathrm{span}\{e_1, \ldots, e_l\}$. *Then, for every $K \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that, for all $N \geq n$, the solution to*

$$A \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \tilde{\beta}_3 \\ \vdots \\ \tilde{\beta}_K \end{pmatrix} = P_K U^* P_N \begin{pmatrix} f(\rho(1)) \\ f(\rho(2)) \\ f(\rho(3)) \\ \vdots \end{pmatrix}, A = P_K U^* P_N U P_K|_{P_K l^2(\mathbb{N})}, \tag{8}$$

is unique.

Theorem

If

$$\tilde{g}_{K,N} = \sum_{j=1}^{K} \tilde{\beta}_j \varphi_j, \qquad \tilde{f}_{K,N} = \sum_{j=1}^{K} \tilde{\beta}_j \mathcal{F}\varphi_j,$$

then

$$\|g - \tilde{g}_{K,N}\|_{L^2(\mathbb{R}^d)} \leq (1 + C_{K,N})\|P_K^{\perp}\beta\|_{l^2(\mathbb{N})}, \quad \beta = \{\beta_1, \beta_2, \dots\},$$

and

$$\|f - \tilde{f}_{K,N}\|_{L^\infty(\mathbb{R}^d)} \leq (2T)^{d/2}(1 + C_{K,N})\|P_K^{\perp}\beta\|_{l^2(\mathbb{N})},$$

where, for fixed K*, the constant* $C_{K,N} \to 0$ *as* $N \to \infty$*.*

of samples v.s. # of coefficients (Haar wavelet)

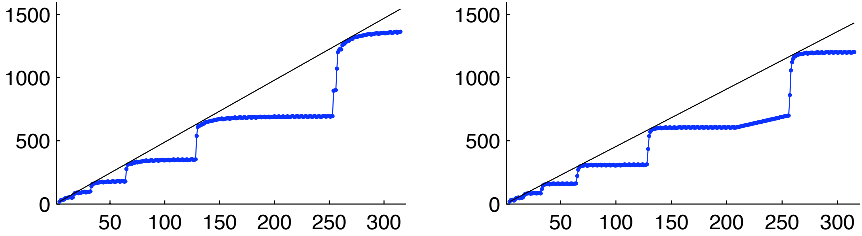
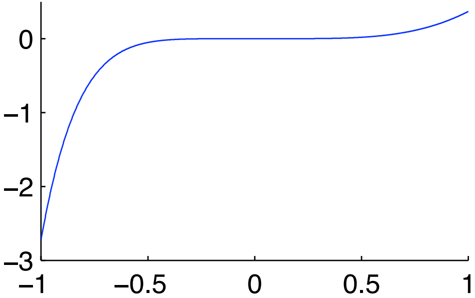


Figure: The left figure shows N (as a function of K) such that $C_{K,N} \leq 1$ together with the functions (in black) $x \mapsto 4.9x$. The right figure shows N such that $C_{K,N} \leq 2$ together with the function $x \mapsto 4.5x$. .

Example:

Let

$$g(t) = t^5 e^{-t}.$$



Approximation via the Shannon Sampling Theorem

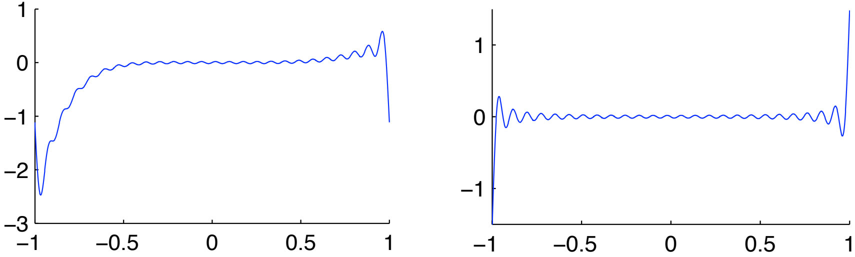


Figure: The figure displays g_N (left) as well as the error $g - g_N$ (right). $N = 25$

Approximation via Generalized Sampling

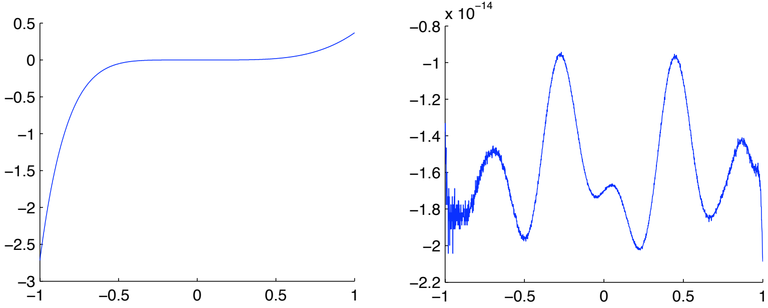


Figure: The figure displays $\tilde{g}_{K,N}$ (left) as well as the error $g - \tilde{g}_{K,N}$ (right) for $K = 12$ and $N = 25$.

(i) Are there other ways of approximating (6)?

(ii) Could there be ways of reconstructing, with the same accuracy, but using fewer samples from $\{\zeta_j(x_0)\}$?

Infinite Dimensional Compressed Sensing

Let $\Omega \subset \mathbb{N}$ such that $|\Omega| = m < \infty$ be randomly chosen and let P_Ω denote the projection onto $\mathrm{span}\{e_j\}_{j\in\Omega}$. Now consider the convex (infinite-dimensional) optimization problem

$$\inf_{\eta \in l^1(\mathbb{N})} \|\eta\|_{l^1(\mathbb{N})} \; : \; P_\Omega \begin{pmatrix} \zeta_1(x_0) \\ \zeta_2(x_0) \\ \zeta_3(x_0) \\ \vdots \end{pmatrix} = P_\Omega \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \end{pmatrix}. \tag{9}$$

(i) How do we randomly choose Ω?It does not make sense to choose Ω uniformly from the whole set $\mathbb{N}$.

(iii) What if we chose an $N \in \mathbb{N}$ and choose $\Omega \subset \{1, \ldots, N\}$ uniformly at random with $|\Omega| = m < N$? But how big must N be?

(iii) If η is a solution to (9) (note that we may not have uniqueness) what is the error $\|\eta - \beta\|_{l^2(\mathbb{N})}$, and how does it depend on the choice of Ω? In particular, how big must m be. (Note that we must have the extra assumption that $\beta \in l^1(\mathbb{N})$.)

Infinite Dimensional Compressed Sensing

The solution to problem (9) cannot be computed explicitly because it is infinite-dimensional, and thus an approximation must be computed instead. For $M \in \mathbb{N}$, consider the optimization problem

$$\inf_{\eta \in P_M l^1(\mathbb{N})} \|\eta\|_{l^1(\mathbb{N})} \; : \; P_\Omega \begin{pmatrix} \zeta_1(x_0) \\ \zeta_2(x_0) \\ \zeta_3(x_0) \\ \vdots \end{pmatrix} = P_\Omega \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} P_M \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_M \end{pmatrix}. \tag{10}$$

(i) If $\tilde{\eta}_M = \{\eta_1, \dots, \eta_M\}$ is a minimizer of (10), what is the behavior of $\tilde{\eta}_M$ as $M \to \infty$? Moreover, what happens to the error $\|\tilde{\eta}_M - \beta\|_{l^2(\mathbb{N})}$ as $M \to \infty$?

(ii) Observe that M cannot be too small, since then (10) may not have a solution. However, (11) is feasible for all sufficiently large M.

The Semi-Infinite-Dimensional Model

We are given an $M \in \mathbb{N}$ and for $x_0 = \sum_{k=1}^{\infty} \beta_k \varphi_k \in \mathcal{H}$ we have that

$$\mathrm{supp}(x_0) = \{j \in \mathbb{N} : \beta_j \neq 0\} = \Delta \subset \{1, \ldots, M\}$$

We will choose only finitely many of the samples $\{\zeta_j(x_0)\}_{j \in \mathbb{N}}$, in particular, we will choose a set $\Omega \subset \{1, \ldots, N\}$ of size m uniformly at random.

- How large must N be?
- How large must m be to recover x_0 with high probability? Moreover, if $m = N$ will we then get perfect recovery with probability one?

The Full Infinite-Dimensional Model

In the full infinite dimensional model we consider the problem of recovering a vector $y_0 = \sum_{k=1}^{\infty} \alpha_k \varphi_k \in \mathcal{H}$ where

$$y_0 = x_0 + h, \qquad h = \sum_{k=1}^{\infty} c_k \varphi_k,$$

$$\mathrm{supp}(x_0) = \Delta \subset \{1, \ldots, M\}, \qquad \mathrm{supp}(h) = \{1, \ldots, \infty\},$$

where we have some estimate on $\sum_{k=1}^{\infty} |c_k|$. In other words, we do not know the support of h.

Consider the optimization problem

$$\inf_{\eta \in P_M l^1(\mathbb{N})} \|\eta\|_{l^1(\mathbb{N})} \; : \; P_\Omega \begin{pmatrix} f(\rho(1)) \\ f(\rho(2)) \\ f(\rho(3)) \\ \vdots \end{pmatrix} = P_\Omega \begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} P_M \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_M \end{pmatrix}. \tag{11}$$

Experiment

Consider the function $f \in L^2(\mathbb{R})$ defined by $f = \mathcal{F}g$

$$g(t) = \sum_{k=1}^{L} \alpha_k \psi_k(t) + \cos(2\pi t)\chi_{[\frac{1}{2},\frac{9}{16}]}(t), \qquad L = 200,$$

where $|\{\alpha_k : \alpha_k \neq 0\}| = 25$, and the task is to reconstruct f from its point samples. Define, for $N \in \mathbb{N}$ and N odd, the function

$$f_N(t) = \sum_{k=-(N-1)/2}^{(N-1)/2} f(k\epsilon)\mathrm{sinc}\left(\frac{t+k\epsilon}{\epsilon}\right), \qquad \epsilon = 0.5$$

Experiment

Define also the functions

$$\tilde{f}_{N,K}(t) = \sum_{k=1}^{K} \tilde{\beta}_k \mathcal{F}\psi_k(t), \qquad \gamma_{N,m,M}(t) = \sum_{k=1}^{M} \eta_k \mathcal{F}\psi_k(t), \quad (12)$$

where $\tilde{\beta} = \{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$ is the solution to equation (8), and U is defined as in (7) (with the Haar wavelets $\{\psi_k\}_{k\in\mathbb{N}}$ as the basis). And also let $\eta = \{\eta_1, \ldots, \eta_M\}$ be a solution to (11) where $\Omega \subset \{1, \ldots, N\}$ is chosen uniformly at random with $|\Omega| = m$.

Results

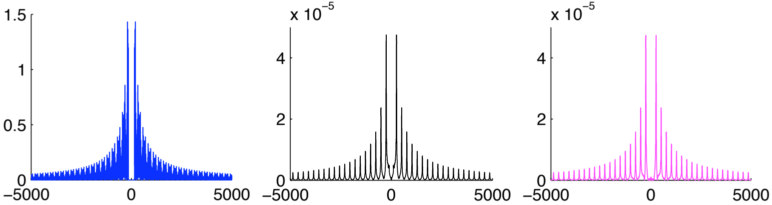


Figure: The figure displays the errors $|f_N - f|$ (left), $|\tilde{f}_{N,K} - f|$ (middle), $|\gamma_{N,m,M} - f|$ (right), for $N = 601, K = 200, m = 230,\ M = 650$. Note that $\gamma_{N,m,M}$ requires only thirty eight percent of the samples.

Theorem

(H'10) Let $U \in \mathcal{B}(\mathcal{H})$ be an isometry. Suppose that for $M \in \mathbb{N}$ we have $\Delta \subset \{1, \dots, M\}$, and $x_0 \in l^1(\mathbb{N})$ such that $\mathrm{supp}(x_0) = \Delta$. Let, for $\epsilon > 0$, the integers m and N be chosen such that

$$\|P_M U^* P_N U P_M - P_M\| \leq \left(4\sqrt{\log_2\left(4N\sqrt{|\Delta|}/m\right)}\right)^{-1}, \tag{13}$$

$$\max_{|\Gamma|=|\Delta|, \Gamma\subset\{1,\dots,M\}} \|P_M P_\Gamma^\perp U^* P_N U P_\Gamma\| \leq \frac{1}{8\sqrt{|\Delta|}}, \tag{14}$$

$$m \geq C \cdot N \cdot \mu^2(U) \cdot |\Delta| \cdot \left(\log\left(\epsilon^{-1}\right) + 1\right) \cdot \log\left(MN\sqrt{|\Delta|}/m\right), \qquad \mu(U) = \sup_{i,j\in\mathbb{N}} |\langle Ue_j, e_i\rangle|. \tag{15}$$

for some universal constant C. Let $\Omega \subset \{1, \dots, N\}$ be chosen uniformly at random with $|\Omega| = m$. If $\xi \in \mathcal{H}$ satisfies

$$\|\xi\|_{l^1} = \inf_{\eta\in\mathcal{H}} \{\|\eta\|_{l^1} : P_\Omega U P_M \eta = P_\Omega U x_0\},$$

then, with probability exceeding $1 - \epsilon$, we have that ξ is unique and $\xi = x_0$. If $m = N$ then ξ is unique and $\xi = x_0$ with probability one.

Corollary

Let $U \in \mathbb{C}^{n\times n}$ be an isometry. Let $\Delta \subset \{1,\dots,n\}$, and $x_0 \in \mathbb{C}^n$ such that $\mathrm{supp}(x_0) = \Delta$. Let, for $\epsilon > 0$, the integer m be chosen such that

$$m \geq C \cdot n \cdot \mu^2(U) \cdot |\Delta| \cdot \left(\log\left(\epsilon^{-1}\right) + 1\right) \cdot \log(n), \qquad (16)$$

for some universal constant C. Let $\Omega \subset \{1,\dots,n\}$ be chosen uniformly at random with $|\Omega| = m$. If $\xi \in \mathcal{H}$ satisfies

$$\|\xi\|_{l^1} = \inf_{\eta\in\mathbb{C}^n}\{\|\eta\|_{l^1} : P_\Omega U\eta = P_\Omega U x_0\},$$

then, with probability exceeding $1-\epsilon$, we have that ξ is unique and $\xi = x_0$.

Theorem

(H'10) Let $U \in \mathcal{B}(\mathcal{H})$ be an isometry. Suppose that for $M \in \mathbb{N}$ we have $\Delta \subset \{1, \dots, M\}$, and $x_0, h \in l^1(\mathbb{N})$ such that $\mathrm{supp}(x_0) = \Delta$. Define $y_0 = x_0 + h$. Let, for $\epsilon > 0$, the integers m and N be chosen according to (13) and also

$$\max_{|\Gamma|=|\Delta|, \Gamma \subset \{1,\dots,M\}} \|P_\Gamma^\perp U^* P_N U P_\Gamma\| \leq \frac{1}{8\sqrt{|\Delta|}},$$

$$m \geq C \cdot N \cdot \mu^2(U) \cdot |\Delta| \cdot \left(\log\left(\epsilon^{-1}\right) + 1\right) \cdot \log\left(\frac{\Theta N \sqrt{|\Delta|}}{m}\right),$$

$$\Theta = \left| \left\{ i \in \mathbb{N} : \max_{\substack{\Gamma_1 \subset \{1,\dots,M\}, |\Gamma_1| = |\Delta| \\ \Gamma_2 \subset \{1,\dots,N\}}} \|P_{\Gamma_1} U^* P_{\Gamma_2} U e_i\| > \frac{m}{4N\sqrt{|\Delta|}} \right\} \right|.$$

for some universal constant C. Let $\Omega \subset \{1, \dots, N\}$ be chosen uniformly at random with $|\Omega| = m$.

Theorem

If $\xi \in \mathcal{H}$ satisfies

$$\|\xi\|_{l^1} = \inf_{\eta \in \mathcal{H}} \{\|\eta\|_{l^1} : P_\Omega U \eta = P_\Omega U y_0\},$$

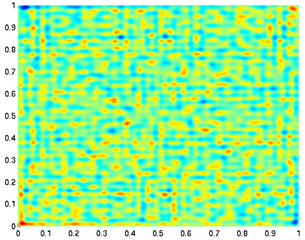
then, with probability exceeding $1 - \epsilon$, we have that

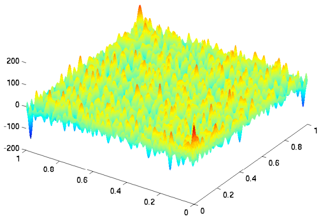
$$\|\xi - y_0\| \leq \left(\frac{20N}{m} + 11 + \frac{m}{N} \right) \|h\|_{l^1}. \tag{17}$$

If $m = N$ then (17) is true with probability one.

Infinite Resolution Image

Let x_0 denote the infinite resolution image:

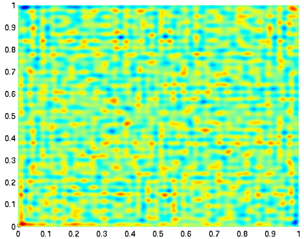


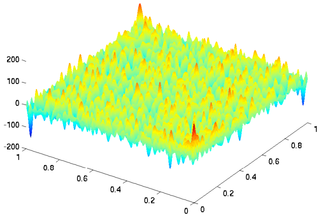


In particular,

$$g = \sum_{j=1}^{\infty} \alpha_j \varphi_j, \quad \varphi_j(x,y) = \sin(kx)\sin(ly), \quad x_0 = \{\alpha_1, \alpha_2, \dots\}$$

Infinite Resolution Image





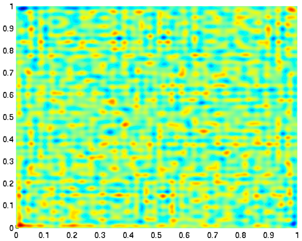
$$|\{\alpha_j : \alpha_j \neq 0\}| = 70,$$

$$\alpha_j = 0, \qquad j > 700.$$

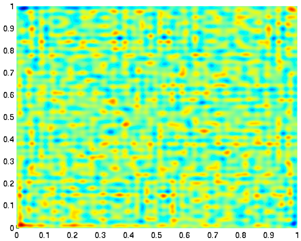
Classical MRI Reconstruction

$$g(t) = \epsilon \sum_{n=-\infty}^{\infty} (Fg)(n\epsilon)\, e^{2\pi i n \epsilon t}, \qquad g_N(t) = \epsilon \sum_{n=-N}^{N} (Fg)(n\epsilon)\, e^{2\pi i n \epsilon t}$$

Original



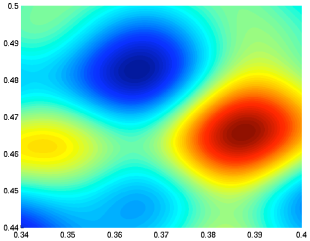
Reconstruction (501 by 501)



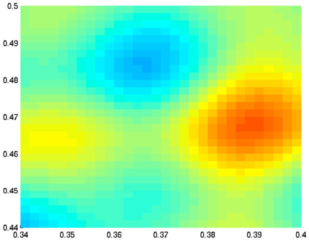
Classical MRI Reconstruction (enlarged)

$$g(t) = \epsilon \sum_{n=-\infty}^{\infty} (Fg)(n\epsilon)\, e^{2\pi i n\epsilon t}, \qquad g_N(t) = \epsilon \sum_{n=-N}^{N} (Fg)(n\epsilon)\, e^{2\pi i n\epsilon t}$$

Original



Reconstruction (501 by 501)

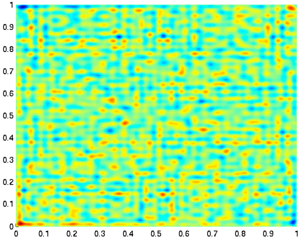


Finite Dim Comp Sens Reconstruction

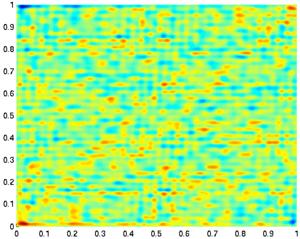
Solve

$$\min_{x} \|x\|_{TV}, \qquad P_{\Omega} U_{dft} x = P_{\Omega} y, \quad |\Omega| = 501^2/2$$

Original



Reconstruction



Sampling

Choose $\epsilon > 0$ ($\epsilon = 0.5$), and consider the grid

$$\epsilon\mathbb{Z} \times \epsilon\mathbb{Z}.$$

Choose a bijection $\rho : \mathbb{N} \to \epsilon\mathbb{Z} \times \epsilon\mathbb{Z}$. Form the infinite matrix

$$U = \begin{pmatrix} F\varphi_1(\rho(1)) & F\varphi_2(\rho(1)) & F\varphi_3(\rho(1)) & \dots \\ F\varphi_1(\rho(2)) & F\varphi_2(\rho(2)) & F\varphi_3(\rho(2)) & \dots \\ F\varphi_1(\rho(3)) & F\varphi_2(\rho(3)) & F\varphi_3(\rho(3)) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

Choose $N \in \mathbb{N}$ ($N = 15000$). Randomly choose a set $\Omega = \{\omega_1, \dots \omega_m\} \subset \{1, \dots, N\}$ such that $|\Omega| = m = 500$. Let $y = \{Fg(\rho(\omega_1), \dots, Fg(\rho(\omega_m))\}$. Then

$$y = P_\Omega U x_0.$$

Recovery

Solve

$$\inf_x \|x\|_{l^1}, \qquad P_\Omega U x = P_\Omega U x_0,$$

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>> norm(x - x_0) = 3.2959e-08
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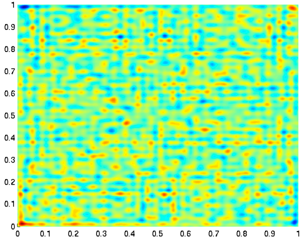
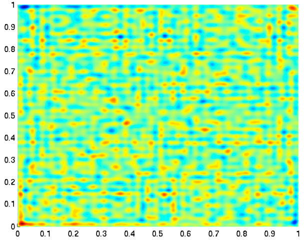
Original



Reconstruction



Comparison